Scalar QED on the lattice

$$S = \int d^4x \left\{ \frac{1}{4e^2} F_{\mu\nu}^2 + |D_\mu \varphi|^2 + m^2 |\varphi|^2 + \frac{\lambda}{4} |\varphi|^4 \right\}$$

Field tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

Covariant der. $D_\mu = \partial_\mu + i A_\mu$

Gauge transf. $A'_\mu = A_\mu - \partial_\mu \lambda$

$\varphi' = e^{i\lambda} \varphi$

$F'_{\mu\nu} = \partial_\mu (A_\nu - \partial_\nu \lambda) - \partial_\nu (A_\mu - \partial_\mu \lambda) = F_{\mu\nu}$

$D'_\mu \varphi' = (\partial_\mu + i A_\mu - i \partial_\mu \lambda)\, e^{i\lambda} \varphi = e^{i\lambda} D_\mu \varphi$

Goal: discretize on the lattice while keeping gauge invariance!

Gauge transf. (continuum)

$$A'_\mu = A_\mu - \partial_\mu \lambda \qquad F'_{\mu\nu} = \partial_\mu (A_\nu - \partial_\nu \lambda) - \partial_\nu (A_\mu - \partial_\mu \lambda) = F_{\mu\nu}$$

$$\varphi' = e^{i\lambda} \varphi \qquad D'_\mu \varphi' = (\partial_\mu + iA_\mu - i\partial_\mu \lambda)\, e^{i\lambda} \varphi = e^{i\lambda} D_\mu \varphi$$

Goal: discretize on the lattice while keeping gauge invariance.

1st attempt   Let's try $A'_\mu = A_\mu - \partial^f_\mu \lambda$ $\varphi' = e^{i\lambda}\varphi$ $D_\mu \varphi = (\partial^f_\mu + iA_\mu)\varphi$

Important!

$$\partial^f_\mu [e^{i\lambda}\varphi](x) \neq i e^{i\lambda(x)} \partial^f_\mu \lambda(x) \varphi(x) + e^{i\lambda(x)} \partial^f_\mu \varphi(x)$$

$$D'_\mu \varphi'(x) = (\partial^f_\mu + iA_\mu - i\partial^f_\mu \lambda)\, e^{i\lambda} \varphi(x)$$

$$= \partial^f_\mu [e^{i\lambda}\varphi](x) - i e^{i\lambda(x)} \partial^f_\mu \lambda(x) \varphi(x) + i e^{i\lambda(x)} A_\mu(x) \varphi(x)$$

$$= \frac{e^{i\lambda(x+ae_\mu)} \varphi(x+ae_\mu) - e^{i\lambda(x)} \varphi(x)}{a} - i e^{i\lambda(x)} \partial^f_\mu \lambda(x) \varphi(x) + i e^{i\lambda(x)} A_\mu(x) \varphi(x)$$

$$\neq e^{i\lambda(x)} D_\mu \varphi(x)$$

Also $|D'_\mu \varphi'|^2 \neq |D_\mu \varphi|^2$ is not gauge invariant!

FAILED!

Gauge transf.
(Continuum)

$A'_\mu = A_\mu - \partial_\mu \lambda$

$\varphi' = e^{i\lambda}\varphi$

$F'_{\mu\nu} = \partial_\mu(A_\nu - \partial_\nu\lambda) - \partial_\nu(A_\mu - \partial_\mu\lambda) = F_{\mu\nu}$

$D'_\mu\varphi' = (\partial_\mu + iA_\mu - i\partial_\mu\lambda)\, e^{i\lambda}\varphi = e^{i\lambda} D_\mu\varphi$

Goal: discretize on the lattice while keeping gauge invariance.

2nd attempt

Let's try $\varphi' = e^{i\lambda}\varphi$ $\quad D_\mu\varphi = (\partial^f_\mu + iA_\mu)\varphi$

Let's impose $D'_\mu\varphi' = e^{i\lambda}D_\mu\varphi$ and find A'_μ

$$D'_\mu\varphi'(x) = (\partial^f_\mu + iA'_\mu)\, e^{i\lambda}\varphi(x) = \frac{1}{a}\left\{e^{i\lambda(x+ae_\mu)}\varphi(x+ae_\mu) - \underline{\underline{e^{i\lambda(x)}\varphi(x)}}\right\} + i e^{i\lambda(x)} A'_\mu(x)\varphi(x)$$

$\|$ require

$$e^{i\lambda(x)} D_\mu\varphi(x) = e^{i\lambda(x)}(\partial^f_\mu + iA_\mu)\varphi(x) = \frac{e^{i\lambda(x)}}{a}\left\{\varphi(x+ae_\mu) - \underline{\underline{\varphi(x)}}\right\} + i e^{i\lambda(x)} A_\mu(x)\varphi(x)$$

Solve for $A'_\mu(x)$:

$$i e^{i\lambda(x)} A'_\mu(x)\varphi(x) = i e^{i\lambda(x)} A_\mu(x)\varphi(x) + \frac{e^{i\lambda(x)}}{a}\varphi(x+ae_\mu) - \frac{e^{i\lambda(x+ae_\mu)}}{a}\varphi(x+ae_\mu)$$

$$A'_\mu(x) = A_\mu(x) + i\,\frac{e^{i[\lambda(x+ae_\mu)-\lambda(x)]} - 1}{a}\,\frac{\varphi(x+ae_\mu)}{\varphi(x)}$$

Transformation of A depends on φ.
FAILED!

Gauge transf. (continuum)

$$A'_\mu = A_\mu - \partial_\mu \lambda \qquad F'_{\mu\nu} = \partial_\mu(A_\nu - \partial_\nu \lambda) - \partial_\nu(A_\mu - \partial_\mu \lambda) = F_{\mu\nu}$$

$$\varphi' = e^{i\lambda}\varphi \qquad D'_\mu \varphi' = (\partial_\mu + iA_\mu - i\partial_\mu \lambda)\, e^{i\lambda}\varphi = e^{i\lambda} D_\mu \varphi$$

Goal: discretize on the lattice while keeping gauge invariance.

Key: understand the geometric meaning of the covariant derivative.

$$\begin{tikzcd}
x & x+ae_\mu \\
\circ & \circ \\
\varphi(x) \arrow[d] & \varphi(x+ae_\mu) \arrow[d] \\
e^{i\lambda(x)}\varphi(x) & e^{i\lambda(x+ae_\mu)}\varphi(x+ae_\mu)
\end{tikzcd}$$

} From the point of view of gauge transformations, it does not make sense to consider $\partial^f_\mu \varphi(x)$

We want to construct a function of $\varphi(x+ae_\mu)$ that transforms like $\varphi(x)$.

$$e^{i\int_0^a ds\, A_\mu(x+se_\mu)}\varphi(x+ae_\mu) \to e^{i\int_0^a ds\, A_\mu(x+se_\mu)}\, e^{-i\int_0^a ds\, \partial_\mu\lambda(x+se_\mu)}\, e^{i\lambda(x+ae_\mu)}\varphi(x+ae_\mu)$$

① $$\left[\begin{aligned} &\frac{d}{ds}\lambda(x+se_\mu) = \\ &= \sum_\rho \partial_\rho\lambda(x+se_\mu)\frac{d}{ds}(x+se_\mu)_\rho \\ &= \sum_\rho \partial_\rho\lambda(x+se_\mu)(e_\mu)_\rho \\ &= \sum_\rho \partial_\rho\lambda(x+se_\mu)\,\delta_{\mu\rho} \\ &= \partial_\mu\lambda(x+se_\mu)\end{aligned}\right]$$

$$\overset{①}{=} e^{i\int_0^a ds\, A_\mu(x+se_\mu)}\, e^{-i\int_0^a ds\, \frac{d}{ds}\lambda(x+se_\mu)}\, e^{i\lambda(x+ae_\mu)}\varphi(x+ae_\mu)$$

$$= e^{i\int_0^a ds\, A_\mu(x+se_\mu)}\, e^{-i[\lambda(x+ae_\mu)-\lambda(x)]}\, e^{i\lambda(x+ae_\mu)}\varphi(x+ae_\mu)$$

$$= e^{i\lambda(x)}\left[e^{i\int_0^a ds\, A_\mu(x+se_\mu)}\right]\varphi(x+ae_\mu)$$

$W(x \to x+av)$ = parallel transporter (or Wilson line) along the straight line from x to $x+av$

$$\equiv \exp\left\{i\int_0^a ds \sum_\mu v_\mu A_\mu(x+sv)\right\} \in U(1)$$

Under gauge transformation

$$W'(x\to x+ae_\mu) = e^{i\lambda(x)}\, W(x\to x+ae_\mu)\, e^{-i\lambda(x+ae_\mu)}$$

$$W'(x\to x+ae_\mu)\,\varphi'(x+ae_\mu) = e^{i\lambda(x)}\, W(x\to x+ae_\mu)\,\varphi(x+ae_\mu)$$

→ generic vector

$$e^{i\int_0^a ds \sum_\mu v_\mu A_\mu(x+sv)}\varphi(x+av) \rightarrow e^{i\int_0^a ds\, v_\mu A_\mu(x+sv)}\, e^{-i\int_0^a ds\, v_\mu \partial_\mu \lambda(x+sv)}\, e^{i\lambda(x+av)}\varphi(x+av)$$

① $$\left[\frac{d}{ds}\lambda(x+sv) = \partial_\mu \lambda(x+sv)\frac{d}{ds}(x+sv)_\mu = v_\mu \partial_\mu \lambda(x+sv)\right]$$

$$\overset{①}{=} e^{i\int_0^a ds\, v_\mu A_\mu(x+sv)}\, e^{-i\int_0^a ds \frac{d}{ds}\lambda(x+sv)}\, e^{i\lambda(x+av)}\varphi(x+av)$$

$$= e^{i\int_0^a ds\, v_\mu A_\mu(x+sv)}\, e^{-i\left[\lambda(x+av)-\lambda(x)\right]}\, e^{i\lambda(x+av)}\varphi(x+av)$$

$$= e^{i\lambda(x)}\, e^{i\int_0^a ds\, v_\mu A_\mu(x+sv)}\varphi(x+av)$$

$W(x \rightarrow x+av)$ = parallel transporter (or Wilson line) along the straight line from x to x+av

$$\equiv \exp\left\{i\int_0^a ds \sum_\mu v_\mu A_\mu(x+sv)\right\} \in U(1)$$

Under gauge transformation

$$W'(x \rightarrow x+av) = e^{i\lambda(x)}\, W(x \rightarrow x+av)\, e^{-i\lambda(x+av)}$$

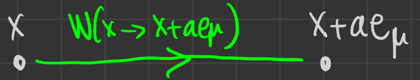
$$W'(x \rightarrow x+av)\varphi(x+av) = e^{i\lambda(x)}\, W(x \rightarrow x+av)\varphi(x+av)$$

Gauge transf. (continuum) $A'_\mu = A_\mu - \partial_\mu \lambda$ $\qquad F'_{\mu\nu} = \partial_\mu(A_\nu - \partial_\nu\lambda) - \partial_\nu(A_\mu - \partial_\mu\lambda) = F_{\mu\nu}$

$\varphi' = e^{i\lambda}\varphi$ $\qquad D'_\mu\varphi' = (\partial_\mu + iA_\mu - i\partial_\mu\lambda)\, e^{i\lambda}\varphi = e^{i\lambda} D_\mu\varphi$

Goal: discretize on the lattice while keeping gauge invariance.

Key: understand the geometric meaning of the covariant derivative.



$$\begin{array}{ccc} \varphi(x) & \qquad & \varphi(x+ae_\mu) \\ \downarrow & & \downarrow \\ e^{i\lambda(x)}\varphi(x) & & e^{i\lambda(x+ae_\mu)}\,\varphi(x+ae_\mu) \end{array}$$

$$W(x\to x+av) \longrightarrow e^{i\lambda(x)}\, W(x\to x+av)\, e^{-i\lambda(x+av)}$$

$$W(x\to x+ae_\mu)\,\varphi(x+ae_\mu) \to e^{i\lambda(x)}\, W(x\to x+ae_\mu)\,\varphi(x)$$

$W(x\to x+ae_\mu)\,\varphi(x+ae_\mu)$ transforms like $\varphi(x)$

"Normal" derivative

$$\partial_\mu\varphi(x) = \frac{d}{da}\,\varphi(x+ae_\mu)\Big|_{a=0}$$

Covariant derivative

$$D_\mu\varphi(x) = \frac{d}{da}\Big[W(x\to x+ae_\mu)\,\varphi(x+ae_\mu)\Big]_{a=0}$$

check: $\displaystyle \frac{d}{da}\Big[e^{i\int_0^a ds A_\mu(x+se_\mu)}\varphi(x+ae_\mu)\Big]_{a=0} = W(x\to x+ae_\mu)\Big[iA_\mu(x+ae_\mu)\,\varphi(x+ae_\mu) + \frac{d}{da}\varphi(x+ae_\mu)\Big]_{a=0} = \big[iA_\mu(x) + \partial_\mu\big]\varphi(x)$

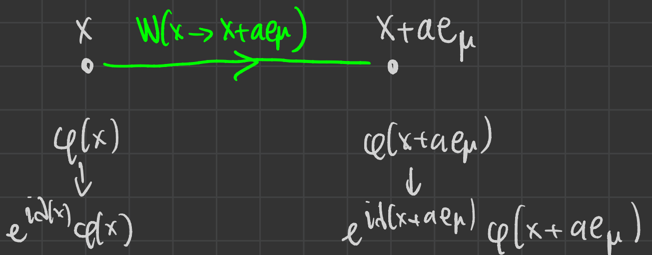
Gauge transf. (continuum)

$$A'_\mu = A_\mu - \partial_\mu \lambda \qquad F'_{\mu\nu} = \partial_\mu(A_\nu - \partial_\nu\lambda) - \partial_\nu(A_\mu - \partial_\mu\lambda) = F_{\mu\nu}$$

$$\varphi' = e^{i\lambda}\varphi \qquad D'_\mu\varphi' = (\partial_\mu + iA_\mu - i\partial_\mu\lambda)\, e^{i\lambda}\varphi = e^{i\lambda} D_\mu\varphi$$

Goal: discretize on the lattice while keeping gauge invariance.

Key: understand the geometric meaning of the covariant derivative.



$$W(x\to x+av) \to e^{i\lambda(x)}\, W(x\to x+av)\, e^{-i\lambda(x+av)}$$

$$W(x\to x+ae_\mu)\,\varphi(x+ae_\mu) \to e^{i\lambda(x)}\, W(x\to x+ae_\mu)\,\varphi(x)$$

$W(x\to x+ae_\mu)\,\varphi(x+ae_\mu)$ transforms like $\varphi(x)$

"Normal" derivative

$$\partial_\mu\varphi(x) = \frac{d}{da}\varphi(x+ae_\mu)\Big|_{a=0} = \lim_{a\to 0}\frac{\varphi(x+ae_\mu)-\varphi(x)}{a}$$

Covariant derivative

$$D_\mu\varphi(x) = \frac{d}{da}\Big[W(x\to x+ae_\mu)\,\varphi(x+ae_\mu)\Big]_{a=0} = \lim_{a\to 0}\frac{W(x\to x+ae_\mu)\,\varphi(x+ae_\mu)-\varphi(x)}{a}$$

If you drop the limit, you get a possible discretization

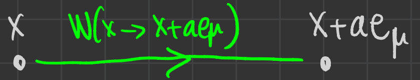
Gauge transf. (continuum)

$$A'_\mu = A_\mu - \partial_\mu \lambda \qquad F'_{\mu\nu} = \partial_\mu(A_\nu - \partial_\nu \lambda) - \partial_\nu(A_\mu - \partial_\mu \lambda) = F_{\mu\nu}$$

$$\varphi' = e^{i\lambda}\varphi \qquad D'_\mu \varphi' = (\partial_\mu + iA_\mu - i\partial_\mu \lambda)\, e^{i\lambda}\varphi = e^{i\lambda} D_\mu \varphi$$

Goal: discretize on the lattice while keeping gauge invariance.

Key: understand the geometric meaning of the covariant derivative.



$$\begin{tikzcd} \varphi(x) \arrow[d] & \varphi(x+ae_\mu) \arrow[d] \\ e^{i\lambda(x)}\varphi(x) & e^{i\lambda(x+ae_\mu)}\varphi(x+ae_\mu) \end{tikzcd}$$

$$W(x \to x+ae_\mu) \to e^{i\lambda(x)}\, W(x \to x+ae_\mu)\, e^{-i\lambda(x+ae_\mu)}$$

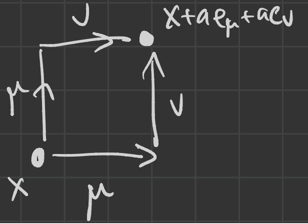
$$W(x \to x+ae_\mu)\,\varphi(x+ae_\mu) \to e^{i\lambda(x)}\, W(x \to x+ae_\mu)\,\varphi(x)$$

$$D^f_\mu \varphi(x) = \frac{W(x \to x+ae_\mu)\varphi(x+ae_\mu) - \varphi(x)}{a}$$ forward discrete covariant derivative

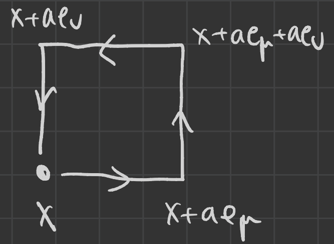
$$D^f_\mu \varphi(x) \to e^{i\lambda(x)} D^f_\mu \varphi(x)$$

Can we write $F_{\mu\nu}$ in terms of parallel transporters?

It must be possible since $F_{\mu\nu} = -i[D_\mu, D_\nu] = -i(D_\mu D_\nu - D_\nu D_\mu)$



Consider the parallel transport along a square in the $\mu\nu$ plane $(\mu \neq \nu)$



$$W(\square) = W(x \to x+ae_\mu)\, W(x+ae_\mu \to x+ae_\mu+ae_\nu) \times$$
$$\times\, W(x+ae_\mu+ae_\nu \to x+ae_\nu)\, W(x+ae_\nu \to x)$$

1) $$W(\underset{x}{\square}) \to e^{i\lambda(x)} W(x \to x+ae_\mu) e^{-i\lambda(x+ae_\mu)}\, e^{i\lambda(x+ae_\mu)} W(x+ae_\mu \to x+ae_\mu+ae_\nu)\, e^{-i\lambda(x+ae_\mu+ae_\nu)} \times$$
$$\times\, e^{i\lambda(x+ae_\mu+ae_\nu)} W(x+ae_\mu+ae_\nu \to x+ae_\nu)\, e^{-i\lambda(x+ae_\nu)}\, e^{i\lambda(x+ae_\nu)} W(x+ae_\nu \to x)\, e^{-i\lambda(x)}$$

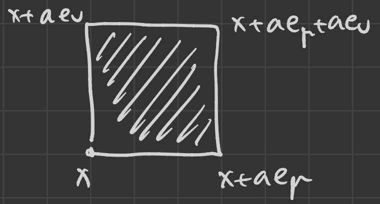
$$= e^{i\lambda(x)}\, W(\underset{x}{\square})\, e^{-i\lambda(x)} = W(\underset{x}{\square})$$

i.e. $W(\square)$ is gauge invariant

2) $$W(\underset{x}{\square}) = e^{i\int_0^a\!\!\int_0^a ds\,dt\, F_{\mu\nu}(x+se_\mu+te_\nu)}$$

← let's prove this

integral of $F_{\mu\nu}$ over the surface



$$= \int_0^a ds\,dt\; F_{\mu\nu}(x+se_\mu+te_\nu) = \int_0^a ds\,dt \;\; (\partial_\mu A_\nu - \partial_\nu A_\mu)(x+se_\mu+te_\nu)$$

$$= \int_0^a ds\,dt \left[\frac{d}{ds} A_\nu(x+se_\mu+te_\nu) - \frac{d}{dt} A_\mu(x+se_\mu+te_\nu) \right]$$

$$= \int_0^a dt \left[\underline{A_\nu(x+ae_\mu+te_\nu)} - \underline{A_\nu(x+te_\nu)} \right] - \int_0^a ds \left[\underline{A_\mu(x+se_\mu+ae_\nu)} - \underline{A_\mu(x+se_\mu)} \right]$$

$$e^{i\int_0^a ds\,dt\, F_{\mu\nu}(x+se_\mu+te_\nu)} = \underline{W(x \to x+ae_\mu)} \;\; \underline{W(x+ae_\mu \to x+ae_\mu+ae_\nu)}$$

$$\times\; \underline{W(x+ae_\nu \to x+ae_\nu+ae_\mu)}^* \;\; \underline{W(x \to x+ae_\nu)}^*$$

General property $W(x+av\to x) = W(x+av \to x+av-av)$

$$= \exp\left\{ i\int_0^a ds \sum_\rho (-v_\rho) A_\rho(x+av-sv)\right\}$$

change of variable
$s'=a-s$ ←

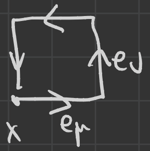
$$= \exp\left\{ -i\int_0^a ds' \sum_\rho v_\rho A_\rho(x+sv)\right\}$$

$$= W(x\to x+av)^*$$

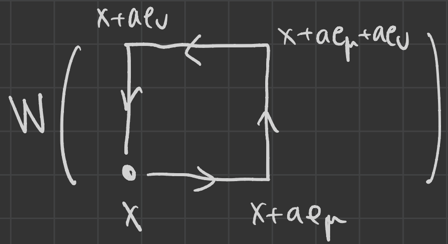
$$e^{i\int_0^a dsdt\, F_{\mu\nu}(x+se_\mu+te_\nu)} = \underline{W(x\to x+ae_\mu)}\ \underline{W(x+ae_\mu \to x+ae_\mu+ae_\nu)}$$

$$\times\ \underline{W(x+ae_\nu \to x+ae_\nu+ae_\mu)}^*\ \underline{W(x\to x+ae_\nu)}^*$$

$$= W(x\to x+ae_\mu)\,W(x+ae_\mu \to x+ae_\mu+ae_\nu)\,W(x+ae_\mu+ae_\nu \to x+ae_\nu)\,W(x+ae_\nu \to x)$$

$= W\Big($  $\Big)$

Consider the parallel transport along a square in the $\mu\nu$ plane $(\mu \neq \nu)$



$$W(\square) = W(x \to x+ae_\mu)\, W(x+ae_\mu \to x+ae_\mu+ae_\nu) \times$$
$$\times\, W(x+ae_\mu+ae_\nu \to x+ae_\nu)\, W(x+ae_\nu \to x)$$

1) $$W(\square_x) \to e^{i\lambda(x)} W(x \to x+ae_\mu) e^{-i\lambda(x+ae_\mu)}\, e^{i\lambda(x+ae_\mu)} W(x+ae_\mu \to x+ae_\mu+ae_\nu)\, e^{-i\lambda(x+ae_\mu+ae_\nu)} \times$$
$$\times\, e^{i\lambda(x+ae_\mu+ae_\nu)} W(x+ae_\mu+ae_\nu \to x+ae_\nu)\, e^{-i\lambda(x+ae_\nu)} e^{i\lambda(x+ae_\nu)} W(x+ae_\nu \to x)\, e^{-i\lambda(x)}$$

$$= e^{i\lambda(x)}\, W(\square_x)\, e^{-i\lambda(x)} = W(\square_x)$$

i.e. $W(\square)$ is gauge invariant

2) $$W(\square_x) = e^{i\int_0^a ds\,dt\, F_{\mu\nu}(x+se_\mu+te_\nu)} = 1 + ia^2 F_{\mu\nu}(x) + O(a^3)$$

$$F_{\mu\nu}(x) = \lim_{a\to 0} \frac{W(\square)-1}{ia^2}$$

3) $$\mathrm{Re}\, W(\square) = 1 - \frac{a^4}{2} F_{\mu\nu}^2(x) + O(a^6)$$ ⟵ let's prove this

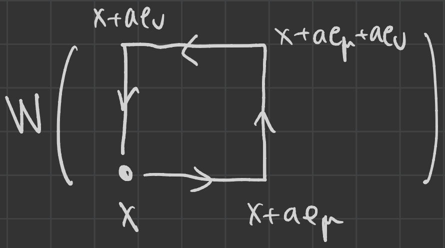
$$W\left(\underset{x}{\square}\right) = e^{i\int_0^a ds\,dt\, F_{\mu\nu}(x+se_\mu+te_\nu)}$$

$$= 1 + \left(i\int_0^a ds\,dt\, F_{\mu\nu}\right) + \frac{1}{2}\left(i\int_0^a ds\,dt\, F_{\mu\nu}\right)^2 + O(a^6)$$

$$\mathrm{Re}\, W\left(\square\right) = 1 - \frac{1}{2}\left\{\int_0^a ds\,dt\, F_{\mu\nu}(x+se_\mu+te_\nu)\right\}^2 + O(a^6) =$$

$$= 1 - \frac{a^4}{2} F_{\mu\nu}^2(x) + O(a^5)$$

Consider the parallel transport along a square in the $\mu\nu$ plane $(\mu \neq \nu)$



$$W\left(\square\right) = W(x \to x+ae_\mu)\, W(x+ae_\mu \to x+ae_\mu+ae_\nu) \times$$
$$\times\, W(x+ae_\mu+ae_\nu \to x+ae_\nu)\, W(x+ae_\nu \to x)$$

1) $$W\left(\underset{x}{\square}\right) \to e^{i\lambda(x)}\, W(x \to x+ae_\mu)\, e^{-i\lambda(x+ae_\mu)}\; e^{i\lambda(x+ae_\mu)}\, W(x+ae_\mu \to x+ae_\mu+ae_\nu)\, e^{-i\lambda(x+ae_\mu+ae_\nu)} \times$$
$$\times\, e^{i\lambda(x+ae_\mu+ae_\nu)}\, W(x+ae_\mu+ae_\nu \to x+ae_\nu)\, e^{-i\lambda(x+ae_\nu)}\, e^{i\lambda(x+ae_\nu)} W(x+ae_\nu \to x)\, e^{-i\lambda(x)}$$

$$= e^{i\lambda(x)}\, W\left(\underset{x}{\square}\right) e^{-i\lambda(x)} = W\left(\underset{x}{\square}\right)$$

i.e. $W\left(\square\right)$ is gauge invariant

2) $$W\left(\underset{x}{\square}\right) = e^{i\int_0^a ds\,dt\, F_{\mu\nu}(x+se_\mu+te_\nu)} = 1 + ia^2 F_{\mu\nu}(x) + O(a^3)$$

$$F_{\mu\nu}(x) = \lim_{a\to 0} \frac{W\left(\square\right) - 1}{ia^2}$$

3) $$\mathrm{Re}\, W\left(\square\right) = 1 - \frac{a^4}{2} F^2_{\mu\nu}(x) + O(a^5)$$

this becomes $O(a)$ only after $\sum_x a^4$

$$\frac{1}{4} F^2_{\mu\nu}(x) = \frac{1}{2a^4}\left\{1 - \mathrm{Re}\, W\left(\underset{x\ \mu}{\square}{}^{\nu}\right)\right\} + O(a)$$

What we've got so far...

$$S_{\text{Cont}} = \int d^4x \left\{ \frac{1}{4e^2} F_{\mu\nu}^2 + |D_\mu\varphi|^2 + m^2|\varphi|^2 + \frac{\lambda}{4}|\varphi|^4 \right\}$$

On smooth field configurations:

$$\lim_{a\to 0} \frac{W(x\to x+ae_\mu)\varphi(x+ae_\mu)-\varphi(x)}{a} \equiv \lim_{a\to 0} D^f_\mu\varphi(x) = D_\mu\varphi(x)$$

$$\lim_{a\to 0} \frac{1}{2a^4}\left[1-\text{Re}\,W\left(\underset{x\quad ae_\mu}{\square}\, ae_\nu\right)\right] = \frac{1}{4}F^2_{\mu\nu}(x)$$

$$S_{\text{lat}} = \sum_{x\in\Lambda}\sum_{\mu\neq\nu}\frac{1}{2e^2}\left[1-\text{Re}\,W\left(\underset{x\quad ae_\mu}{\square}\, ae_\nu\right)\right] + \sum_{x\in\Lambda} a^4\left\{|D^f_\mu\varphi|^2 + m^2|\varphi|^2 + \frac{\lambda}{4}|\varphi|^4\right\}$$

1) On smooth field configurations $\lim_{a\to 0} S_{\text{lat}} = S$

2) S_{lat} depends only on $W(x\to x+ae_\mu)$ and $\varphi(x)$ for $x\in\Lambda$.
In particular S_{lat} depends on A_μ only via W.

3) S_{lat} is invariant under gauge transformations
$$W(x\to x+ae_\mu) \to e^{i\lambda(x)}\,W(x\to x+ae_\mu)\,e^{-i\lambda(x+ae_\mu)}$$
$$\varphi(x) \to e^{i\lambda(x)}\varphi(x)$$

$$S_{lat}(U,\varphi) = \sum_{x\in\Lambda}\sum_{\mu\neq\nu}\frac{1}{2e^2}\left[1-\mathrm{Re}\,W\Big(\underset{x\ \ ae_\mu}{\square}{}^{ae_\nu}\Big)\right] + \sum_{x\in\Lambda} a^4\left\{|D^e_\mu\varphi|^2 + m^2|\varphi|^2 + \frac{\lambda}{4}|\varphi|^4\right\}$$

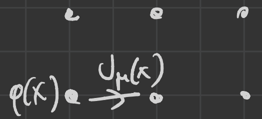
1) On smooth field configurations $\lim_{a\to 0} S_{lat} = S$

2) S_{lat} depends only on $W(x\to x+ae_\mu)$ and $\varphi(x)$ for $x\in\Lambda$.
In particular S_{lat} depends on A_μ only via W.

3) S_{lat} is invariant under gauge transformations
$W(x\to x+ae_\mu)\to e^{i\lambda(x)}\,W(x\to x+ae_\mu)\,e^{-i\lambda(x+ae_\mu)}$
$\varphi(x)\to e^{i\lambda(x)}\varphi(x)$

Now we take this action seriously and we declare that:

- $U_\mu(x) \equiv W(x\to x+e_\mu a)$ with $x\in\Lambda$ [link variables]
- $\varphi(x)$ with $x\in\Lambda$



are the fundamental degrees of freedom of the regularized theory.

[$\varphi(x)_{x\in\Lambda}$ is a sampling of the field $\varphi(x)_{x\in\mathbb{R}^4}$
$U_\mu(x)_{x\in\Lambda}$ is a sort of gauge-covariant sampling of the field $A_\mu(x)_{x\in\mathbb{R}^4}$]

[Also in hard-cutoff regulariz. you throw away information]

In order to define the path-integral of scalar QED, we need

(1) a discretized gauge-invariant action ✓

(2) a gauge-invariant integration measure.

- The scalar field is complex $\varphi(x) = \mathrm{Re}\,\varphi(x) + i\,\mathrm{Im}\,\varphi(x)$

Natural choice $[d\varphi\, d\varphi^*] = \prod_x d\mathrm{Re}\,\varphi(x)\, d\mathrm{Im}\,\varphi(x)$

Under a gauge transformation

$$\varphi'(x) = e^{i\alpha(x)}\varphi(x) = [\cos\alpha(x) + i\sin\alpha(x)]\,[\mathrm{Re}\,\varphi(x) + i\,\mathrm{Im}\,\varphi(x)]$$

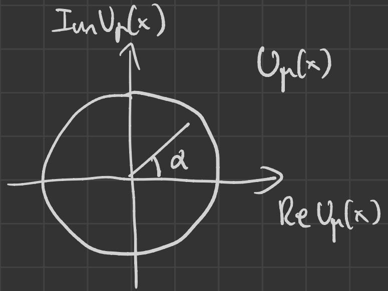
$$\begin{aligned} \mathrm{Re}\,\varphi' &= \cos\alpha\, \mathrm{Re}\,\varphi - \sin\alpha\, \mathrm{Im}\,\varphi \\ \mathrm{Im}\,\varphi' &= \sin\alpha\, \mathrm{Re}\,\varphi + \cos\alpha\, \mathrm{Im}\,\varphi \end{aligned} \quad \Big\| \quad \begin{pmatrix} \mathrm{Re}\,\varphi' \\ \mathrm{Im}\,\varphi' \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \mathrm{Re}\,\varphi \\ \mathrm{Im}\,\varphi \end{pmatrix}$$

$$[d\varphi'] = \prod_x \left| \det \begin{pmatrix} \cos\alpha(x) & -\sin\alpha(x) \\ \sin\alpha(x) & \cos\alpha(x) \end{pmatrix} \right| d\mathrm{Re}\,\varphi(x)\, d\mathrm{Im}\,\varphi = \prod_x d\mathrm{Re}\,\varphi(x)\, d\mathrm{Im}\,\varphi(x) = [d\varphi] \quad ✓$$

In order to define the path-integral of scalar QED, we need

(1) a discretized gauge-invariant action ✓

(2) a gauge-invariant integration measure. ✓



- $U_\mu(x) \in U(1) = \{ z \in \mathbb{C} \text{ s.t. } |z| = 1 \}$

Parametrize $U_\mu(x) = e^{i\alpha_\mu(x)}$ with $\alpha_\mu(x) \in [0, 2\pi)$

$$\int_{U(1)} dU_\mu(x)\, f(U_\mu(x)) = \frac{1}{2\pi} \int_0^{2\pi} d\alpha_\mu(x)\, f(e^{i\alpha_\mu(x)})$$

[rotational-invariant integration measure on the circle]
→ unique up to normalization

Under a gauge transformation $U'_\mu(x) = e^{i\lambda(x)} U_\mu(x) e^{-i\lambda(x+ae_\mu)}$

$$\alpha'_\mu(x) = \alpha_\mu(x) + \lambda(x) - \lambda(x+ae_\mu)$$

$$\int_0^{2\pi} d\alpha'_\mu(x)\, f(e^{i\alpha'(x)}) = \int_{\lambda(x)-\lambda(x+ae_\mu)}^{\lambda(x)-\lambda(x+ae_\mu)+2\pi} d\alpha_\mu(x)\, f(e^{i\alpha'_\mu(x)}) \underset{\substack{\uparrow \\ \text{periodicity of integrand}}}{=} \int_0^{2\pi} d\alpha_\mu(x)\, f(e^{i\alpha'_\mu(x)})$$ ✓

In order to define the path-integral of scalar QED, we need

(1) a discretized gauge-invariant action

$$S_{lat}(U,\varphi) = \sum_{x\in\Lambda} \sum_{\mu\neq\nu} \frac{1}{2e^2}\left[1 - \mathrm{Re}\, W\Big(\underset{x\ \ a\hat{e}_\mu}{\square}{}^{a\hat{e}_\nu}\Big)\right] + \sum_{x\in\Lambda} a^4 \left\{ |D^{f}_{\mu}\varphi|^2 + m^2|\varphi|^2 + \frac{\lambda}{4}|\varphi|^4 \right\}$$

(2) a gauge-invariant integration measure. $[d\varphi d\varphi^*]\,[dU]$

$$[d\varphi d\varphi^*] = \prod_{x} d\mathrm{Re}\,\varphi(x)\, d\mathrm{Im}\,\varphi(x)$$

$$[dU] = \prod_{x\mu} dU_\mu(x)$$

$dU_\mu(x)$ is the rotational-invariant measure on the circle with unit circumference

In a finite lattice, the partition function is finite without gauge fixing!

$$S_{lat}(U,\varphi) \geq \sum_{x\in\Lambda} a^4 m^2 |\varphi|^2 \quad\Rightarrow\quad Z = \int [d\varphi d\varphi^* dU]\, e^{-S_{lat}} \leq \int [d\varphi d\varphi^* dU]\, e^{-\sum_{x\in\Lambda} a^4 m^2 |\varphi|^2} =$$

$$= \left[\prod_{x\mu} \int dU_\mu(x)\right] \left[\prod_{x} \int d\mathrm{Re}\,\varphi(x)\, d\mathrm{Im}\,\varphi(x)\; e^{-a^4 m^2 [\mathrm{Re}\,\varphi(x)]^2}\, e^{-a^4 m^2 [\mathrm{Im}\,\varphi(x)]^2}\right]$$

$$= 1 \cdot \left[\int_{-\infty}^{\infty} dy\; e^{-a^4 m^2 y^2}\right]^{2\left(\frac{L}{a}\right)^2} < +\infty \qquad !$$

	U(1) gauge theory + scalar field	SU(N) gauge theory + scalar field
DOF	$\phi(x) \in \mathbb{C}$ $\quad U_\mu(x) \in U(1)$	$\phi(x) \in \mathbb{C}^N$ $\quad U_\mu(x) \in SU(N)$
Gauge transform.	$\phi(x) \to e^{i\alpha(x)}$ $U_\mu(x) \to e^{i\alpha(x)}\, U_\mu(x)\, e^{i\alpha(x+ae_\mu)}$ $\alpha(x) \in \mathbb{R}$ i.e. $e^{i\alpha(x)} \in U(1)$	$\phi(x) \to \Omega(x)\phi(x)$ $U_\mu(x) \to \Omega(x)\, U_\mu(x)\, \Omega(x+ae_\mu)$ $\Omega(x) \in SU(N)$
Forward covariant derivative	$D^f_\mu \phi = \frac{1}{a}\left\{ U_\mu(x)\phi(x+ae_\mu) - \phi(x) \right\}$	same
Action $S = S_g + S_m$	$S_g = \frac{1}{2e^2} \sum_{x\in\Lambda} \sum_{\mu\neq\nu} \left[1 - \mathrm{Re}\, W\Big(\underset{x \quad ae_\mu}{\square}{}^{ae_\nu} \Big) \right]$	$S_g =$
	$S_M = \sum_{x\in\Lambda} a^4 \left\{ \lvert D^f_\mu \varphi \rvert^2 + m^2 \lvert\varphi\rvert^2 + \frac{\lambda}{4!} \lvert\varphi\rvert^4 \right\}$	same

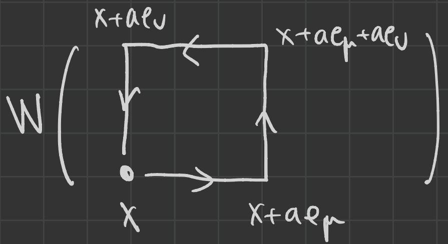
Path-integral measure

$$[d\varphi][d\varphi^*][dU] \equiv \prod_x \left\{ d\mathrm{Re}\,\varphi(x)\, d\mathrm{Im}\,\varphi(x) \prod_\mu dU_\mu(x) \right\}$$

$dU_\mu(x)$ is the rotational-invariant measure on the circle with unit length

$dU_\mu(x)$ is the Haar measure on $SU(N)$

<table>
<tr><th></th><th>U(1) gauge theory + scalar field</th><th>SU(N) gauge theory + scalar field</th></tr>
<tr><td>Gauge invariance</td><td colspan="2">The action and the measure are invariant under gauge transformations</td></tr>
<tr><td>"Naive continuum limit"</td><td>Let $A_\mu(x)\in\mathbb{R}$ and $\varphi(x)\in\mathbb{C}$ be smooth fields in the continuum.
If we identify
$U_\mu(x) = W(x\to x+ae_\mu) = e^{i\int_0^a ds\, A_\mu(x+se_\mu)}$
then
$\lim_{a\to 0} S_{lat}(U,\varphi) =$
$= \int_x \left\{ \frac{1}{4e^2} F_{\mu\nu}^2 + |D_\mu\varphi|^2 + m^2|\varphi^2| + \frac{\lambda}{4}|\varphi|^4 \right\}$</td><td>Let $A_\mu(x)$ be a complex hermitean traceless $N\times N$ matrix, smooth in $x\in\mathbb{R}^4$. Let $\varphi(x)\in\mathbb{C}$ be smooth in $x\in\mathbb{R}^4$
If we identify
$U_\mu(x) = W(x\to x+ae_\mu) = \underrightarrow{P\exp} \int_0^a ds\, A_\mu(x+se_\mu)$
then
$\lim_{a\to 0} S_{lat}(U,\varphi)$
$= \int_x \left\{ \frac{1}{2e^2} \mathrm{tr} F_{\mu\nu}^2 + |D_\mu\varphi|^2 + m^2|\varphi^2| + \frac{\lambda}{4}|\varphi|^4 \right\}$</td></tr>
<tr><td>Finiteness of Z</td><td colspan="2">$Z = \int [d\varphi d\varphi^\dagger dU]\, e^{-S_{lat}(U,\varphi)}$ is finite on a finite lattice</td></tr>
</table>

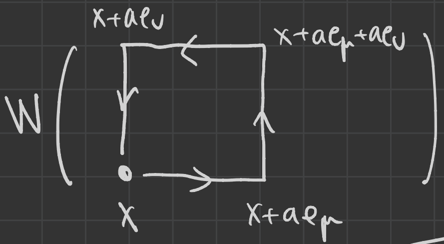


$$= W(x \to x+ae_\mu)\, W(x+ae_\mu \to x+ae_\mu+ae_\nu) \times$$
$$\times\, W(x+ae_\mu+ae_\nu \to x+ae_\nu)\, W(x+ae_\nu \to x)$$

$$W(x \to x+ae_\mu) = \exp\left\{ i \int_0^a ds\, A_\mu(x+se_\mu) \right\} = \exp\left\{ i \int_0^a ds \left[A_\mu(x) + s\, \partial_\mu A_\mu(x) + O(s^2) \right] \right\}$$
$$= \exp\left\{ i\, a A_\mu(x) + \frac{i}{2} a^2\, \partial_\mu A_\mu(x) + O(a^3) \right\}$$

$$W(x+ae_\mu \to x+ae_\mu+ae_\nu) = \exp\left\{ i a A_\nu(x+ae_\mu) + \frac{i}{2} a^2\, \partial_\nu A_\nu(x+ae_\mu) + O(a^3) \right\}$$
$$= \exp\left\{ i a A_\nu(x) + i a^2\, \partial_\mu A_\nu(x) + \frac{i}{2} a^2\, \partial_\nu A_\nu(x) + O(a^3) \right\}$$

$$W(x \to x+ae_\mu)\, W(x+ae_\mu \to x+ae_\mu+a\nu) = \exp\left\{ i a\, (A_\mu + A_\nu)(x) + \frac{i}{2} a^2\, (\partial_\mu A_\mu + \partial_\nu A_\nu)(x) + i a^2\, \partial_\mu A_\nu(x) + O(a^3) \right\}$$



$$= W(x \to x+ae_\mu)\, W(x+ae_\mu \to x+ae_\mu+ae_\nu) \times$$

$$\times\, W(x+ae_\mu+ae_\nu \to x+ae_\nu)\, W(x+ae_\nu \to x)$$

$$= W(x \to x+ae_\mu)\, W(x+ae_\mu \to x+ae_\mu+ae_\nu)$$

$$\times \left[W(x \to x+ae_\nu)\, W(x+ae_\nu \to x+ae_\nu+ae_\mu) \right]^*$$

$$= \left[W(x \to x+ae_\mu)\, W(x+a\mu \to x+ae_\mu+ae_\nu) \right] \times \left[\mu \leftrightarrow \nu \right]^*$$

General property

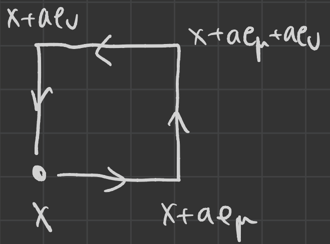
$$W(x+av \to x) = W(x+av \to x+av-av)$$

$$= \exp\left\{ i \int_0^a ds \sum_\rho (-v_\rho)\, A_\rho(x+av-sv) \right\}$$

change of variable $s' = a - s$ ←

$$= \exp\left\{ -i \int_0^a ds' \sum_\rho v_\rho\, A_\rho(x+sv) \right\}$$

$$= W(x \to x+av)^*$$

$W\Big($  $\Big)$

$$= W(x \to x+ae_\mu)\, W(x+ae_\mu \to x+ae_\mu+ae_\nu) \times$$
$$\times\, W(x+ae_\mu+ae_\nu \to x+ae_\nu)\, W(x+ae_\nu \to x)$$

$$= W(x \to x+ae_\mu)\, W(x+ae_\mu \to x+ae_\mu+ae_\nu)$$
$$\times \left[W(x \to x+ae_\nu)\, W(x+ae_\nu \to x+ae_\nu+ae_\mu)\right]^*$$

$$= \left[W(x \to x+ae_\mu)\, W(x+a\mu \to x+ae_\mu+ae_\nu)\right] \times \left[\mu \leftrightarrow \nu\right]^*$$

$$= \exp\left\{ia\,(A_\mu+A_\nu) + \frac{i}{2}a^2(\partial_\mu A_\mu+\partial_\nu A_\nu) - ia^2\,\partial_\mu A_\nu + O(a^3)\right\}$$
$$\exp\left\{-ia\,(A_\mu+A_\nu) - \frac{i}{2}a^2(\partial_\mu A_\mu+\partial_\nu A_\nu) + ia^2\,\partial_\nu A_\mu + O(a^3)\right\}$$
$$= \exp\left\{-ia^2 F_{\mu\nu}(x) + O(a^3)\right\} = 1 - ia^2 F_{\mu\nu}(x) + O(a^3)$$

$$W(x \to x+ae_\mu)\, W(x+ae_\mu \to x+ae_\mu+a\nu) = \exp\left\{ia\,(A_\mu+A_\nu)(x) + \frac{i}{2}a^2(\partial_\mu A_\mu+\partial_\nu A_\nu)(x) + ia^2\,\partial_\mu A_\nu(x) + O(a^3)\right\}$$

Calculation in radial gauge

$$\begin{cases} A^r_\mu(z) = A_\mu(z) - \partial_\mu \partial(z) \\ z_\mu A^r_\mu(x+z) = 0 \quad \forall z \end{cases}$$

$$z_\mu A^r_\mu(x+z) = 0 \iff z_\mu \partial_\mu \partial(x+z) = z_\mu A_\mu(x+z) \qquad \forall z$$

$$\Rightarrow \frac{d}{d\alpha} \partial(x+\alpha z) = z_\mu \partial_\mu \partial(x+\alpha z) = z_\mu A_\mu(x+\alpha z)$$

$$\Rightarrow \partial(x+z) = \partial(x) + \int_0^1 d\alpha \frac{d}{d\alpha} \partial(x+\alpha z) = \partial(x) + \int_0^1 d\alpha \, z_\mu A_\mu(x+\alpha z)$$

Choose $\partial(x) = 0$:

$$\partial(z) = (z-x)_\rho \int_0^1 d\alpha \, A_\rho((1-\alpha)x + \alpha z)$$

$$A^r_\mu(z) = A_\mu(z) - \int_0^1 d\alpha \underset{\substack{\wedge \\ 1=\frac{d\alpha}{d\alpha}}}{} A_\mu((1-\alpha)x+\alpha z) - (z-x)_\rho \int_0^1 d\alpha \, \alpha \, \partial_\mu A_\rho((1-\alpha)x+\alpha z)$$

$$= \int_0^1 d\alpha \, \alpha \frac{d}{d\alpha}\left[A_\mu((1-\alpha)x+\alpha z)\right] - (z-x)_\rho \int_0^1 d\alpha \, \alpha \, \partial_\mu A_\rho((1-\alpha)x+\alpha z)$$

$$= (z-x)_\rho \int_0^1 d\alpha \, \alpha \, F_{\rho\mu}((1-\alpha)x+\alpha z) \qquad \checkmark$$

$$\underset{\mu}{A^r}(x+z) = z_\rho \int_0^1 d\alpha\, \alpha\, F_{\rho\mu}(x+\alpha z)$$

$$= z_\rho \int_0^1 d\alpha \sum_{n=0}^{\infty} \frac{\alpha^{n+1}}{n!} z_{\alpha_1}\cdots z_{\alpha_n} \partial_{\alpha_1}\cdots\partial_{\alpha_n} F_{\rho\mu}(x)$$

$$= z_\rho \sum_{n=0}^{\infty} \frac{1}{(n+1)!} z_{\alpha_1}\cdots z_{\alpha_n} \partial_{\alpha_1}\cdots\partial_{\alpha_n} F_{\rho\mu}(x)$$

$$A^r_\mu(x+ae_\mu+se_\nu) = s \sum_{n=0}^{\infty} \frac{1}{(n+1)!} (ae_\mu+se_\nu)_{\alpha_1}\cdots(ae_\mu+se_\nu)_{\alpha_n} \partial_{\alpha_1}\cdots\partial_{\alpha_n} F_{\nu\mu}(x)$$

$$\int_0^a ds\, A^r_\mu(x+se_\nu) = \sum_{n=0}^{\infty} \frac{a^{n+2}}{(n+2)!} \partial_\nu^n F_{\nu\mu}(x)$$

$$W(\Box) = e^{i\int_0^a ds\, A^r_\nu(x+ae_\mu+se_\nu) - i\int_0^a ds\, A^r_\mu(x+ae_\nu+se_\mu)}$$

$$= e^{ia\int_0^1 ds\, A^r_\nu(x+a(e_\mu+se_\nu)) - (\mu\leftrightarrow\nu)}$$

$$= e^{ia^2\int_0^1 ds\, (e_\mu+se_\nu)_\rho \int_0^1 d\alpha\, \alpha\, F_{\rho\nu}(x+\alpha a(e_\mu+se_\nu)) - (\mu\leftrightarrow\nu)}$$

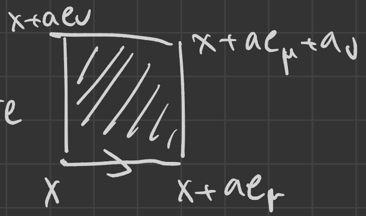
$F_{\rho\nu}(x) + a(e_\mu+se_\nu$

$x = (x_0, x_1, x_2, x_3)$ $\qquad\qquad$ $\mu = 0, \quad \nu = 1$

[repeated indices are not summed]

$\mu \neq \nu$

$D_{\mu\nu} =$ full square



$$e^{i\int_{D_{\mu\nu}} dz_\mu dz_\nu F_{\mu\nu}(z)}\Big|_{\substack{z_\rho = x_\rho \\ \text{if } \rho\neq\mu,\nu}} = \underline{W(x\to x+ae_\mu)}\ \underline{W(x+ae_\mu \to x+ae_\mu+ae_\nu)}$$

$$\times \left[\underline{W(x\to x+ae_\nu)}\ \underline{W(x+ae_\nu+ae_\mu\to x+ae_\nu)}\right]^*$$

$$\begin{bmatrix}\mu=0\\ \nu=1\end{bmatrix} \int_{D_{01}} dz_0 dz_1 F_{01}(z)\Big|_{\substack{z_2=x_2\\ z_3=x_3}} = \int_{x_0}^{x_0+a} dz_0 \int_{x_1}^{x_1+a} dz_1 \,(\partial_0 A_1 - \partial_1 A_0)(z_0, z_1, x_2, x_3)$$

$$= \int_{x_1}^{x_1+a} dz_1 \left[\underline{A_1(x_0+a, z_1, x_2, x_3)} - \underline{A_1(x_0, z_1, x_2, x_3)}\right]$$

$$- \int_{x_0}^{x_0+a} dz_0 \left[\underline{A_0(z_0, x_1+a, x_2, x_3)} - \underline{A_0(z_0, x_1, x_2, x_3)}\right]$$

$$= \underline{\int_0^a ds\, A_0(x+se_0)} + \int_0^a ds\, \underline{A_1(x+ae_0+se_1)} - \int_0^a ds\, \underline{A_0(x+ae_1+se_0)}$$

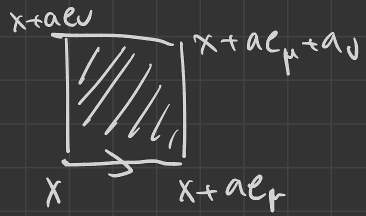
$$- \int_0^a \underline{ds\, A_1(x+se_1)}$$

$x = (x_0, x_1, x_2, x_3)$ $\mu = 0, \quad \nu = 1$

[repeated indices are <u>not</u> summed]

$\mu \neq \nu$

$D_{\mu\nu}$ = full square



$$e^{i\int_{D_{\mu\nu}} dx_\mu dx_\nu F_{\mu\nu}} = \underline{W(x\to x+ae_\mu)}\;\underline{W(x+ae_\mu \to x+ae_\mu+ae_\nu)}$$

$$\times\left[\underline{W(x\to x+ae_\nu)}\;\underline{W(x+ae_\nu+ae_\mu \to x+ae_\nu)}\right]^*$$

General property

$$W(x+av\to x) = W(x+av \to x+av-av)$$

$$= \exp\left\{ i\int_0^a ds \sum_\rho (-v_\rho) A_\rho(x+av-sv) \right\}$$

change of variable $\leftarrow$

$s' = a-s$

$$= \exp\left\{ -i\int_0^a ds' \sum_\rho v_\rho A_\rho(x+sv) \right\}$$

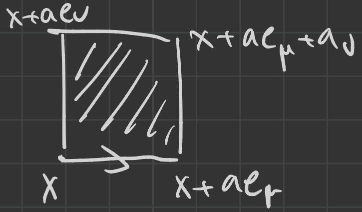
$$= W(x\to x+av)^*$$

use this here

$x = (x_0, x_1, x_2, x_3)$ $\mu = 0, \quad \nu = 1$

$\mu \neq \nu$

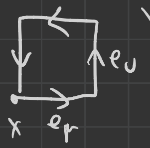
$D_{\mu\nu}$ = full square



[repeated indices are not summed]

$\mu \neq \nu$

$$e^{i\int_{D_{\mu\nu}} dz_\mu dz_\nu F(z)}\Bigg|_{\substack{z_\rho = x_\rho \\ \text{if } \rho \neq \mu,\nu}} = W(x \to x+ae_\mu)\, W(x+ae_\mu \to x+ae_\mu+ae_\nu)\, W(x+ae_\mu+ae_\nu \to x+ae_\nu)\, W(x+ae_\nu \to x)$$

$= W\Big($  $\Big)$